

LAURENT SERIES EXPANSIONS OF MULTIPLE ZETA-FUNCTIONS OF EULER-ZAGIER TYPE AT INTEGER POINTS

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ABSTRACT. We give explicit expressions (or at least an algorithm of obtaining such expressions) of the coefficients of the Laurent series expansions of the Euler-Zagier multiple zeta-functions at any integer points. The main tools are the Mellin-Barnes integral formula and the harmonic product formulas. The Mellin-Barnes integral formula is used in the induction process on the number of variables, and the harmonic product formula is used to show that the Laurent series expansion outside the domain of convergence can be obtained from that inside the domain of convergence.

1. INTRODUCTION AND THE STATEMENT OF MAIN RESULTS

The Euler-Zagier r -ple zeta-function is defined by ([5], [15])

$$(1.1) \quad \zeta_r(\mathbf{s}) = \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} \cdots \sum_{n_r=1}^{\infty} n_1^{-s_1} (n_1 + n_2)^{-s_2} \cdots (n_1 + \cdots + n_r)^{-s_r}$$

(where $\mathbf{s} = (s_1, \dots, s_r) \in \mathbb{C}^r$) in the domain of its absolute convergence, which is

$$(1.2) \quad \mathcal{D}_r = \{\mathbf{s} \in \mathbb{C}^r \mid \Re(s(j, r)) > r - j + 1 \ (1 \leq j \leq r)\},$$

where $s(j, r) = s_j + s_{j+1} + \cdots + s_r$ ([7, Theorem 3]). Special values $\zeta_r(\mathbf{m})$ ($\mathbf{m} = (m_1, \dots, m_r) \in \mathbb{Z}^r$) of (1.1) in this domain are called multiple zeta values (MZV), and have been studied extensively.

It is known that (1.1) can be continued meromorphically to the whole space \mathbb{C}^r (Akiyama, Egami and Tanigawa [1], Zhao [16]). Therefore the behavior of (1.1) around the points $\mathbf{m} \in \mathbb{Z}^r$ outside the domain \mathcal{D}_r is also of great interest. This direction of research was initiated by Akiyama, Egami and Tanigawa [1], and then pursued further by subsequent mathematicians (Akiyama and Tanigawa [2], Komori [6], Sasaki [13] [14], and the second-named author [11]).

In this paper, to understand the behavior of (1.1) around the integer points more closely, we give Laurent series expansions for the Euler-Zagier r -ple zeta-function at integer points. When $r = 1$, the function $\zeta_1(s)$ is nothing but the Riemann zeta-function, and this function has, at $m \in \mathbb{Z}$, the following Taylor or Laurent series expansion:

$$(1.3) \quad \zeta(s) = \begin{cases} \sum_{n=0}^{\infty} \frac{1}{n!} \zeta^{(n)}(m) (s-m)^n & (m \neq 1), \\ \frac{1}{s-1} + \sum_{n=0}^{\infty} \gamma_n (s-1)^n & (m = 1), \end{cases}$$

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where γ_n is the n -th Stieltjes constant (or generalized Euler constant)[†] and $\zeta^{(n)}(s)$ is the n -th derivative of the Riemann zeta-function. The 0-th Stieltjes constant $\gamma_0 = \gamma$ is the well-known Euler constant, and Stieltjes constants have been studied by several authors; see [3, p.164]. Note that for $m > 1$, $\zeta^{(n)}(m)$ is given by the series $\sum_{k=1}^{\infty} (-\log k)^n / k^m$. (In this paper, we define $(-\log 1)^0 = 0^0 = 1$.)

The same type of expansions holds in the multivariable case. Generally, for a function $f(\mathbf{s})$ and $(n_1, \dots, n_r) \in \mathbb{Z}_{\geq 0}$, we denote by $f^{(n_1, \dots, n_r)}(\mathbf{s})$ the (partial) derivative $(\partial^{n_1} / \partial s_1^{n_1}) \cdots (\partial^{n_r} / \partial s_r^{n_r}) f(\mathbf{s})$. When $\mathbf{m} \in \mathcal{D}_r$, obviously

(1.4)

$$\zeta_r(\mathbf{s}) = \sum_{n_1=0}^{\infty} \cdots \sum_{n_r=0}^{\infty} \frac{1}{n_1! \cdots n_r!} \zeta_r^{(n_1, \dots, n_r)}(m_1, \dots, m_r) (s_1 - m_1)^{n_1} \cdots (s_r - m_r)^{n_r}$$

when \mathbf{s} is close to \mathbf{m} .

Also, we define the r -ple (n_1, \dots, n_r) -th Stieltjes constant $\gamma_{(n_1, \dots, n_r)}$ by the following formula which is valid when \mathbf{s} is close to $(1, \dots, 1)$:

$$(1.5) \quad \zeta_r(s_1, \dots, s_r) = \left(\prod_{k=2}^r \frac{1}{s_k + \cdots + s_r - (r - k + 1)} \right) \left\{ \frac{1}{s_1 + \cdots + s_r - r} + \sum_{n_1=0}^{\infty} \cdots \sum_{n_r=0}^{\infty} \gamma_{(n_1, \dots, n_r)} (s_1 - 1)^{n_1} \cdots (s_r - 1)^{n_r} \right\}.$$

When $r = 1$, (regarding the empty product as 1) we find that (1.5) is reduced to the second formula of (1.3), so $\gamma_{(n_1)} = \gamma_{n_1}$.

In Section 3, we will prove the above (1.5), which we call the Laurent series expansion at $(s_1, \dots, s_r) = (1, \dots, 1)$ of the Euler-Zagier r -ple zeta-function (see Lemma 3.1).

Remark 1.1. In the present paper we use the term "Laurent series expansion" in the above extended way, that is, the linear factors in the denominator and in the numerator are not necessarily the same. The standard definition of Laurent series expansion is more restricted; see [4, Chapter I, Section 5, pp.88-90] or [12, Erstes Kapitel, Sektion 20, pp.68-70].

Remark 1.2. It is not clear what is the most suitable definition of multiple Stieltjes constants. See Remark 4.4 at the end of Section 4.

The first main theorem in this paper is as follows.

Theorem 1.3. *Let r, m_1, \dots, m_r be positive integers, and put $\mathbf{m} = (m_1, \dots, m_r)$. Then the coefficients in the Laurent series of $\zeta_r(\mathbf{s})$ at $\mathbf{s} = \mathbf{m}$ can be given explicitly by using $\gamma_{(n_1, \dots, n_k)}$ and $\zeta_k^{(l_1, \dots, l_k)}(q_1, \dots, q_k)$ where $1 \leq k \leq r$, $(n_1, \dots, n_k), (l_1, \dots, l_k) \in (\mathbb{Z}_{\geq 0})^k$ and $(q_1, \dots, q_k) \in \mathcal{D}_k \cap (\mathbb{Z}_{\geq 1})^k$.*

Here we give some comments on the meaning of this theorem.

(i) When $r = 1$, this theorem is trivial, because the assertion is nothing but (1.3) for $m \geq 1$. Similarly, in the general r -ple case for $r \geq 1$, the theorem is trivial by (1.4) when $\mathbf{m} \in \mathcal{D}_r \cap (\mathbb{Z}_{\geq 1})^r$, and follows directly from (1.5) when

[†]Some authors call $(-1)^n n! \gamma_n$ the n -th Stieltjes constant.

$\mathbf{m} = (1, \dots, 1)$. The main point of the theorem lies in the cases when $\mathbf{m} \in (\mathbb{Z}_{\geq 1})^r \setminus (\mathcal{D}_r \cup (1, \dots, 1))$, which happens only when $r \geq 2$.

(ii) When $\mathbf{m} \in (\mathbb{Z}_{\geq 1})^r \setminus (\mathcal{D}_r \cup (1, \dots, 1))$, we understand the meaning of the "Laurent series" at $\mathbf{s} = \mathbf{m}$ in the extended sense mentioned in Remark 1.1. More strictly, when $m_j > 1$ and $m_{j+1} = \dots = m_r = 1$ ($1 \leq j \leq r-1$), then the Laurent series at $\mathbf{s} = \mathbf{m}$ is of the form of a fraction, whose denominator is the product of linear factors $(s(k, r) - (r - k + 1))$ ($j+1 \leq k \leq r$), and its numerator is a Taylor series with respect to $(s_1 - m_1), \dots, (s_r - m_r)$. An example will be given in Example 3.3 at the end of Section 3. The term "coefficients in the Laurent series" in the statement of the theorem means the coefficients in this Taylor series. As can be seen in Example 3.3, our proof of Theorem 1.3 gives an algorithm of obtaining the coefficients explicitly.

After mentioning some preparatory results in Section 2, we will prove this theorem in Section 3. Note that similar to the one variable case, $\zeta_k^{(l_1, \dots, l_k)}(q_1, \dots, q_k)$ also has the series expression

$$\sum_{n_1=1}^{\infty} \dots \sum_{n_k=1}^{\infty} \frac{(-\log n_1)^{l_1} (-\log(n_1 + n_2))^{l_2} \dots (-\log(n_1 + \dots + n_k))^{l_k}}{n_1^{q_1} (n_1 + n_2)^{q_2} \dots (n_1 + \dots + n_k)^{q_k}}$$

for $(l_1, \dots, l_k) \in (\mathbb{Z}_{\geq 0})^k$ and $(q_1, \dots, q_k) \in \mathcal{D}_k$.

In Section 4, we will consider the coefficients in the Laurent series of $\zeta_r(\mathbf{s})$ under a certain restriction on the variables. Under this restriction, these coefficients can be explicitly given by only using γ_n and $\zeta_k^{(l_1, \dots, l_k)}(q_1, \dots, q_k)$. It implies that we can eliminate the role of $\gamma_{(n_1, \dots, n_k)}$ from Theorem 1.3 under this restriction.

In Section 5, we will consider the Laurent series expansions at $\mathbf{m} \in \mathbb{Z}^r \setminus (\mathbb{Z}_{\geq 1})^r$. Since $\zeta_r(\mathbf{s})$ is singular on the hyperplanes

$$s_r = 1, \quad s_{r-1} + s_r = 2, 1, 0, -2, -4, -6, \dots,$$

and

$$s(j, r) = l \quad (l \in \mathbb{Z}, l \leq r - j + 1)$$

for $1 \leq j \leq r-2$ ([1, Theorem 1]), the points $\mathbf{m} \in \mathbb{Z}^r \setminus (\mathbb{Z}_{\geq 1})^r$ are frequently on these singular hyperplanes, and in many cases are the points of indeterminacy. All of the aforementioned previous studies on non-positive integer points encountered this obstacle, and those studies discussed the limit values of $\zeta_r(\mathbf{s})$ when \mathbf{s} approaches \mathbf{m} along various ways. In particular, the second-named author [11] obtained a rather general result in which it allows a lot of flexibility how to approach the limit points, though he did not arrive at the Laurent series expansions. An example of this result will be mentioned just after Corollary 5.2.

We give Laurent series expansions at the points belonging to $\mathbb{Z}^r \setminus (\mathbb{Z}_{\geq 1})^r$, which is the second main theorem in this paper. Define

$$(1.6) \quad M_r(\mathbf{m}) = M_r(m_1, \dots, m_r) := \max\{r - j - (m_j + \dots + m_r) \mid 1 \leq j \leq r\}.$$

For $l \geq 2$ and $z_l \in \mathbb{C}$, let

$$(1.7) \quad F_{z_l}(\mathbf{s}) := \frac{\Gamma(s_l + z_l)}{\Gamma(s_l)} \zeta_{l-1}(s_1, \dots, s_{l-2}, s_{l-1} + s_l + z_l) \quad \text{for } \mathbf{s} = (s_1, \dots, s_l).$$

Then the result is the following

Theorem 1.4. *Let $\mathbf{m} \in \mathbb{Z}^r \setminus (\mathbb{Z}_{\geq 1})^r$. Then the coefficients of the Laurent series of $\zeta_r(\mathbf{s})$ at $\mathbf{s} = \mathbf{m}$ can be given explicitly in terms of the coefficients used in Theorem 1.3, $\zeta^{(n)}(m)$ ($m \leq 0, n \geq 0$), and integrals*

$$\int_{(M_l(\mathbf{k})+1-\eta)} F_{z_l}^{(n_1, \dots, n_l)}(\mathbf{k}) \Gamma(-z_l) \zeta(-z_l) dz_l$$

where $2 \leq l \leq r$, $(n_1, \dots, n_l) \in (\mathbb{Z}_{\geq 0})^l$, $\mathbf{k} \in \mathbb{Z}^l \setminus (\mathbb{Z}_{\geq 1})^l$, $0 < \eta < 1$, and the path of integration is the vertical line $\Re z_l = M_l(\mathbf{k}) + 1 - \eta$.

A big difference from Theorem 1.3 is that, here, the coefficients may include some integrals. This theorem will be proved in the first half of Section 5, and in the second half of Section 5, we will discuss some cases when we may ignore the contribution of integral terms.

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2. PRELIMINARIES

In the following sections, we use in the induction process the following key formula proved by the first-named author (see [8, (12.7)] or [9, (4.4)]):

$$\begin{aligned} (2.1) \quad & \zeta_r(s_1, \dots, s_r) \\ &= \frac{1}{s_r - 1} \zeta_{r-1}(s_1, \dots, s_{r-2}, s_{r-1} + s_r - 1) \\ &+ \sum_{k_r=0}^{M-1} \binom{-s_r}{k_r} \zeta_{r-1}(s_1, \dots, s_{r-2}, s_{r-1} + s_r + k_r) \zeta(-k_r) \\ &+ \frac{1}{\Gamma(s_r)} I(s_1, \dots, s_r; M - \eta), \end{aligned}$$

where $r \geq 2$, M is a positive integer, η is a small positive number, and

$$\begin{aligned} & I(s_1, \dots, s_r; \alpha) \\ &= \frac{1}{2\pi i} \int_{(\alpha)} \Gamma(s_r + z_r) \Gamma(-z_r) \zeta_{r-1}(s_1, \dots, s_{r-2}, s_{r-1} + s_r + z_r) \zeta(-z_r) dz_r, \end{aligned}$$

whose path of integration is the vertical line $\Re z_r = \alpha$. The formula (2.1), which was proved by using the classical Mellin-Barnes integral formula, is valid in the region where the above integral is convergent.

For any points $\mathbf{m} \in \mathbb{Z}^r$, there exists a sufficiently large M such that the integral of (2.1) is analytic at \mathbf{m} (see Lemma 3.2). Hence this formula tells us the behavior of $\zeta_r(s_1, \dots, s_r)$ around the point $\mathbf{s} = \mathbf{m}$ from the information on the behavior of ζ_{r-1} .

Another key formula which we use is the so-called harmonic product formula. This is used to show that the Laurent series expansion outside the domain of convergence can be obtained from that inside the domain of convergence. The harmonic product formula is obtained by just decomposing the summation. For

example, the product of the Euler-Zagier double zeta-function and the Riemann zeta-function can be decomposed as follows:

(2.2)

$$\begin{aligned}
& \zeta_2(s_1, s_2)\zeta(s_3) \\
&= \sum_{n_1 < n_2} \sum_{0 < m} \frac{1}{n_1^{s_1} n_2^{s_2} m^{s_3}} \\
&= \left(\sum_{n_1 < n_2 < m} + \sum_{n_1 < n_2 = m} + \sum_{n_1 < m < n_2} + \sum_{n_1 = m < n_2} + \sum_{m < n_1 < n_2} \right) \frac{1}{n_1^{s_1} n_2^{s_2} m^{s_3}} \\
&= \zeta_3(s_1, s_2, s_3) + \zeta_2(s_1, s_2 + s_3) + \zeta_3(s_1, s_3, s_2) + \zeta_2(s_1 + s_3, s_2) + \zeta_3(s_3, s_1, s_2).
\end{aligned}$$

Similarly we obtain

(2.3)

$$\begin{aligned}
& \zeta(s_1)\zeta_2(s_2, s_3) \\
&= \zeta_3(s_1, s_2, s_3) + \zeta_2(s_1 + s_2, s_3) + \zeta_3(s_2, s_1, s_3) + \zeta_2(s_2, s_1 + s_3) + \zeta_3(s_2, s_3, s_1).
\end{aligned}$$

The same method can be applied to the decomposition of more general product of two Euler-Zagier multiple zeta-functions. For example, as a direct generalization of (2.2), we have

$$\begin{aligned}
(2.4) \quad & \zeta_{r-1}(s_1, \dots, s_{r-1})\zeta(s_r) \\
&= \zeta_r(s_1, \dots, s_r) + \zeta_r(s_1, \dots, s_{r-2}, s_r, s_{r-1}) + \dots + \zeta_r(s_r, s_1, \dots, s_{r-1}) \\
&+ \zeta_{r-1}(s_1, \dots, s_{r-2}, s_{r-1} + s_r) + \zeta_{r-1}(s_1, \dots, s_{r-3}, s_{r-2} + s_r, s_{r-1}) + \dots \\
&+ \zeta_{r-1}(s_1 + s_r, s_2, \dots, s_{r-1}).
\end{aligned}$$

The most general form of the decomposition can be written as

$$\begin{aligned}
(2.5) \quad & \zeta_j(s_1, \dots, s_j)\zeta_{r-j}(s_{j+1}, \dots, s_r) \\
&= \zeta_r(s_1, \dots, s_r) + \zeta_r(s_1, \dots, s_{j-1}, s_{j+1}, s_j, s_{j+2}, s_{j+3}, \dots, s_r) + \dots \\
&+ \zeta_r(s_{j+1}, s_1, s_2, \dots, s_{j-1}, s_j, s_{j+2}, \dots, s_r) + \dots \\
&+ \zeta_r(s_{j+1}, \dots, s_r, s_1, \dots, s_j) \\
&+ (\text{the sum of } \zeta_l \text{ } (l < r))
\end{aligned}$$

for $1 \leq j \leq r-1$.

3. THE LAURENT SERIES EXPANSION AT POSITIVE INTEGER POINTS

The main aim of this section is to prove Theorem 1.3. First, we determine the order of the pole of the Euler-Zagier multiple zeta-function. For $\delta > 0$, let

$$\mathcal{E}_j(\delta) = \{(s_1, \dots, s_j) \in \mathbb{C}^j \mid \Re s_l > 1 - \delta \text{ } (1 \leq l \leq j)\}.$$

Lemma 3.1. *For each $j \in \mathbb{Z}_{\geq 1}$, there exists a function $h_j(s_1, \dots, s_{j+1})$, analytic in the region $\mathcal{E}_{j+1}(\delta_j)$ with a sufficiently small positive constant δ_j (depending only on j), such that, for any $r \in \mathbb{Z}_{\geq 2}$, the identity*

(3.1)

$$\zeta_r(\mathbf{s})$$

$$\begin{aligned}
&= \frac{1}{(s_r - 1)(s_{r-1} + s_r - 2) \cdots (s_2 + \cdots + s_r - (r - 1))} \zeta(s_1 + \cdots + s_r - (r - 1)) \\
&+ \frac{1}{(s_r - 1)(s_{r-1} + s_r - 2) \cdots (s_3 + \cdots + s_r - (r - 2))} h_1(s_1, s_2) \\
&+ \frac{1}{(s_r - 1)(s_{r-1} + s_r - 2) \cdots (s_4 + \cdots + s_r - (r - 3))} h_2(s_1, s_2, s_3) \\
&+ \cdots \\
&+ \frac{1}{s_r - 1} h_{r-2}(s_1, \dots, s_{r-1}) \\
&+ h_{r-1}(s_1, \dots, s_r),
\end{aligned}$$

holds in the region $\mathcal{E}_r(\delta_{r-1})$, especially at any point $\mathbf{m} \in (\mathbb{Z}_{\geq 1})^r$.

To prove Lemma 3.1, we use the part (ii) of the following lemma.

Lemma 3.2. (i) For any $\mathbf{m} \in \mathbb{Z}^r$, if we choose $M \geq M_r(\mathbf{m}) + 1$ (where $M_r(\mathbf{m})$ is defined in (1.6)) and $0 < \eta < 1$, the integral $I(s_1, \dots, s_r; M - \eta)$ is analytic at \mathbf{m} .

(ii) In particular, $I(s_1, \dots, s_r; 1 - \eta)$ is analytic at any point $\mathbf{m} \in (\mathbb{Z}_{\geq 1})^r$. In fact, $I(s_1, \dots, s_r; 1 - \eta)$ is analytic in the region $\mathcal{E}_r(1/r)$.

(Proof of Lemma 3.2.) (i) Let $\mathcal{F}_r(M, \eta)$ be the set of $(s_1, \dots, s_r) \in \mathbb{C}^r$ satisfying

$$(3.2) \quad \Re(s_j + \cdots + s_r) > r - j - M + \eta \quad (1 \leq j \leq r).$$

By [8, Section 12], the integral $I(s_1, \dots, s_r; M - \eta)$ is analytic on $\mathcal{F}_r(M, \eta)$. Consider the case $\mathbf{s} = \mathbf{m}$. We see that $\mathbf{s} = \mathbf{m} \in \mathcal{F}_r(M, \eta)$ if

$$(3.3) \quad M > r - j - (m_j + \cdots + m_r) + \eta \quad (1 \leq j \leq r).$$

It is obvious that under the conditions $M \geq M_r(\mathbf{m}) + 1$ and $0 < \eta < 1$, the inequality (3.3) holds. Hence $\mathbf{m} \in \mathcal{F}_r(M, \eta)$ holds for $M \geq M_r(\mathbf{m}) + 1$.

(ii) If $\mathbf{m} \in (\mathbb{Z}_{\geq 1})^r$, then $M_r(\mathbf{m}) \leq -1$, so we can choose $M = 1$ in assertion (i). This implies the first half of assertion (ii). When $\Re s_l > 1 - \delta$ ($1 \leq l \leq r$), then (3.2) with $M = 1$ is satisfied if $(r - j + 1)(1 - \delta) > r - j - 1 + \eta$, that is, $2 - \eta > (r - j + 1)\delta$ for $1 \leq j \leq r$. This is valid if we choose any δ satisfying $0 < \delta < (2 - \eta)/r$, especially $\delta = 1/r$. The second half of (ii) hence follows. \square

(Proof of Lemma 3.1.) We use the induction on r . First, we consider the case $r = 2$. By (2.1) with $M = 1$, we have

$$\zeta_2(s_1, s_2) = \frac{1}{s_2 - 1} \zeta(s_1 + s_2 - 1) - \frac{1}{2} \zeta(s_1 + s_2) + \frac{1}{\Gamma(s_2)} I(s_1, s_2; 1 - \eta),$$

because $\zeta(0) = -1/2$. From Lemma 3.2 (ii), the last term is analytic at $\mathbf{s} \in (\mathbb{Z}_{\geq 1})^2$. Furthermore the second term is also analytic at $\mathbf{s} \in (\mathbb{Z}_{\geq 1})^2$, since $\zeta(s)$ has only one pole at $s = 1$. Hence the case $r = 2$ is done. (The sum of the second term and the third term gives $h_1(s_1, s_2)$, with $0 < \delta_1 \leq 1/2$.)

Now we assume that (3.1) holds for $r - 1$, that is, we have

$$(3.4) \quad \zeta_{r-1}(s_1, \dots, s_{r-2}, s'_{r-1})$$

$$\begin{aligned}
&= \frac{1}{(s'_{r-1} - 1) \cdots (s_2 + \cdots + s_{r-2} + s'_{r-1} - (r-2))} \zeta(s_1 + \cdots + s'_{r-1} - (r-2)) \\
&+ \frac{1}{(s'_{r-1} - 1) \cdots (s_3 + \cdots + s_{r-2} + s'_{r-1} - (r-3))} h_1(s_1, s_2) \\
&+ \cdots \\
&+ \frac{1}{s'_{r-1} - 1} h_{r-3}(s_1, \dots, s_{r-2}) \\
&+ h_{r-2}(s_1, \dots, s'_{r-1}).
\end{aligned}$$

On the other hand, we use (2.1) with $M = 1$ to obtain

$$\begin{aligned}
(3.5) \quad \zeta_r(s_1, \dots, s_r) &= \frac{1}{s_r - 1} \zeta_{r-1}(s_1, \dots, s_{r-2}, s_{r-1} + s_r - 1) - \frac{1}{2} \zeta_{r-1}(s_1, \dots, s_{r-2}, s_{r-1} + s_r) \\
&+ \frac{1}{\Gamma(s_r)} I(s_1, \dots, s_r; 1 - \eta).
\end{aligned}$$

Substituting (3.4) to (3.5) with $s'_{r-1} = s_{r-1} + s_r - 1$, we have

$$\begin{aligned}
&\zeta_r(s_1, \dots, s_r) \\
&= \frac{1}{s_r - 1} \left\{ \frac{1}{(s_{r-1} + s_r - 2) \cdots (s_2 + \cdots + s_r - (r-1))} \zeta(s_1 + \cdots + s_r - (r-1)) \right. \\
&+ \frac{1}{(s_{r-1} + s_r - 2) \cdots (s_3 + \cdots + s_r - (r-2))} h_1(s_1, s_2) \\
&+ \cdots \\
&+ \frac{1}{(s_{r-1} + s_r - 2)} h_{r-3}(s_1, \dots, s_{r-2}) \\
&\left. + h_{r-2}(s_1, \dots, s_{r-2}, s_{r-1} + s_r - 1) \right\} \\
&- \frac{1}{2} \zeta_{r-1}(s_1, \dots, s_{r-2}, s_{r-1} + s_r) + \frac{1}{\Gamma(s_r)} I(s_1, \dots, s_r; 1 - \eta).
\end{aligned}$$

Since $(m_1, \dots, m_{r-2}, m_{r-1} + m_r) \in \mathcal{D}_{r-1}$ holds for $\mathbf{m} \in (\mathbb{Z}_{\geq 1})^r$, it is clear that $\zeta_{r-1}(s_1, \dots, s_{r-2}, s_{r-1} + s_r)$ is analytic at these points. Furthermore it follows from Lemma 3.2 (ii) that the last term is also analytic. The term containing h_{r-2} can be written as

$$\frac{h_{r-2}(s_1, \dots, s_{r-2}, s_{r-1} + s_r - 1) - h_{r-2}(s_1, \dots, s_{r-2}, s_{r-1})}{s_r - 1} + \frac{h_{r-2}(s_1, \dots, s_{r-1})}{s_r - 1}.$$

The first term here is analytic at $\mathbf{m} \in (\mathbb{Z}_{\geq 1})^r$. (When $s_r = 1$, this term is to be understood as the derivative of h_{r-2} with respect to the last variable.) Hence putting

$$\begin{aligned}
&h_{r-1}(s_1, \dots, s_r) \\
&:= \frac{h_{r-2}(s_1, \dots, s_{r-2}, s_{r-1} + s_r - 1) - h_{r-2}(s_1, \dots, s_{r-1})}{s_r - 1} \\
&- \frac{1}{2} \zeta_{r-1}(s_1, \dots, s_{r-2}, s_{r-1} + s_r)
\end{aligned}$$

$$+ \frac{1}{\Gamma(s_r)} I(s_1, \dots, s_r; 1 - \eta),$$

which is analytic in $\mathcal{E}_r(\delta_{r-1})$ for a sufficiently small $\delta_{r-1} > 0$, we obtain Lemma 3.1. \square

(*Proof of (1.5).*) This is immediate from the formula (3.1) of Lemma 3.1, with expanding the Riemann zeta factor to the Laurent series, and h_j factors to the Taylor series. \square

Next we prove Theorem 1.3.

(*Proof of Theorem 1.3.*) We use the induction on r . When $r = 1$, it follows from (1.3) that Theorem 1.3 holds. We assume that Theorem 1.3 holds when the number of variables is $1, 2, \dots, r-1$, and we prove that Theorem 1.3 holds for r .

Let $\mathbf{m} \in (\mathbb{Z}_{\geq 1})^r$. We define the case (C_j) as

$$(3.6) \quad (C_j) : \begin{cases} m_r > 1 & \text{if } j = r, \\ m_j > 1, m_{j+1} = m_{j+2} = \dots = m_r = 1 & \text{if } 1 \leq j \leq r-1. \end{cases}$$

When $m_r > 1$, since $\mathbf{m} \in \mathcal{D}_r$ holds, the series (1.1) is absolutely convergent at \mathbf{m} . Therefore $\zeta_r(\mathbf{s})$ has the Taylor series expansion (1.4) at \mathbf{m} , so we are done in the case (C_r) .

Next we consider the case (C_j) , $1 \leq j \leq r-1$. We use the (down-going) induction on j . Let $1 \leq j \leq r-1$, assume that Theorem 1.3 holds in the cases $(C_{j+1}), \dots, (C_r)$, and we prove Theorem 1.3 for (C_j) . Our tool is the formula (2.5). The first term on the right-hand side of (2.5) is that we want to expand. Coefficients in the Laurent series of the left-hand side and of the terms of ζ_l ($l < r$) can be given explicitly by using $\gamma_{(n_1, \dots, n_k)}$ and $\zeta_k^{(l_1, \dots, l_k)}(q_1, \dots, q_k)$ by the assumption of induction on r . Coefficients in the Laurent series of the terms of ζ_r except the first term on the right-hand side can be also given explicitly by using $\gamma_{(n_1, \dots, n_k)}$ and $\zeta_k^{(l_1, \dots, l_k)}(q_1, \dots, q_k)$, by the assumption of induction on j , because in these terms, s_j is located at the l -th element of the r -tuple where $l > j$. Hence in this case, Theorem 1.3 holds.

Therefore by induction, we obtain Theorem 1.3 for (C_j) ($j = 1, \dots, r-1$). Finally, the only remaining case $\mathbf{m} = (1, \dots, 1)$ is implied by (1.5). Thus we complete the proof of Theorem 1.3. (We can also see the fact mentioned in comment (ii) just after the statement of Theorem 1.3, by analyzing the above proof a little more carefully.) \square

Example 3.3. Here we explain the procedure given in the proof of Theorem 1.3, by describing the case $r = 3$, $\mathbf{m} = (2, 1, 1)$. Assume s_1 is close to 2, and s_2, s_3 are close to 1. We first use (2.3):

$$(3.7) \quad \begin{aligned} \zeta_3(s_1, s_2, s_3) &= \zeta(s_1)\zeta_2(s_2, s_3) - \zeta_2(s_1 + s_2, s_3) \\ &\quad - \zeta_3(s_2, s_1, s_3) - \zeta_2(s_2, s_1 + s_3) - \zeta_3(s_2, s_3, s_1). \end{aligned}$$

Since s_1 is close to 2 and $s_1 + s_3$ is close to 3, the last two terms on the right-hand side are in the domain of absolute convergence, so can be expanded by (1.4).

In particular, these are $O(1)$. Next, we apply (2.2) to the third term on the right-hand side:

$$(3.8) \quad \begin{aligned} \zeta_3(s_2, s_1, s_3) &= \zeta(s_2, s_1)\zeta(s_3) - \zeta_2(s_2, s_1 + s_3) \\ &\quad - \zeta_3(s_2, s_3, s_1) - \zeta_2(s_2 + s_3, s_1) - \zeta_3(s_3, s_2, s_1). \end{aligned}$$

Since s_1 is close to 2, all terms but the first one on the right-hand side are in the domain of absolute convergence, hence can be expanded by (1.4) and $O(1)$. The first term can be expanded by (1.3) and (1.4), and can be written as

$$\zeta(s_2, s_1)\zeta(s_3) = \zeta_2(s_2, s_1) \left(\frac{1}{s_3 - 1} + O(1) \right).$$

Therefore

$$(3.9) \quad \zeta_3(s_2, s_1, s_3) = \frac{\zeta_2(s_2, s_1)}{s_3 - 1} + O(1).$$

To the second term on the right-hand side of (3.7), we apply the simplest harmonic product formula

$$(3.10) \quad \zeta(s_1)\zeta(s_2) = \zeta_2(s_1, s_2) + \zeta_2(s_2, s_1) + \zeta(s_1 + s_2)$$

to obtain

$$\zeta_2(s_1 + s_2, s_3) = \zeta(s_1 + s_2)\zeta(s_3) - \zeta_2(s_3, s_1 + s_2) - \zeta(s_1 + s_2 + s_3).$$

The second and the third terms on the right-hand side are in the domain of absolute convergence, and so

$$(3.11) \quad \zeta_2(s_1 + s_2, s_3) = \frac{\zeta(s_1 + s_2)}{s_3 - 1} + O(1).$$

Finally, since (s_2, s_3) is close to $(1, 1)$, we use (1.5) to obtain

$$(3.12) \quad \zeta(s_1)\zeta_2(s_2, s_3) = \zeta(s_1) \frac{1}{s_3 - 1} \left(\frac{1}{s_2 + s_3 - 2} + A(s_2, s_3) \right),$$

where

$$A(s_2, s_3) = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \gamma_{(n_1, n_2)} (s_2 - 1)^{n_1} (s_3 - 1)^{n_2}.$$

By the above argument it is clear that $\zeta_3(s_1, s_2, s_3)$ can be expanded to the Laurent series around the point $(2, 1, 1)$, and especially

$$(3.13) \quad \begin{aligned} \zeta_3(s_1, s_2, s_3) &= \frac{\zeta(s_1)}{(s_3 - 1)(s_2 + s_3 - 2)} \\ &\quad + \frac{\zeta(s_1)A(s_2, s_3) - \zeta(s_1 + s_2) - \zeta_2(s_2, s_1)}{s_3 - 1} + O(1), \end{aligned}$$

where the terms on the numerators are all holomorphic there. Expanding the numerators and the $O(1)$ term to the Taylor series, we obtain an example of comment (ii) after the statement of Theorem 1.3.

4. THE LAURENT SERIES EXPANSION AT POSITIVE INTEGER POINTS UNDER A CERTAIN ADDITIONAL RESTRICTION

In this section, we consider the Laurent series expansion of (1.1) under a certain additional restriction on the variables.

Definition 4.1. Let $\mathbf{m} = (m_1, \dots, m_r) \in (\mathbb{Z}_{\geq 1})^r$. Let h be the number of m_j which is equal to 1, and denote those m_j s by m_{a_1}, \dots, m_{a_h} . By the *restricted Laurent series expansion* of $\zeta_r(\mathbf{s})$ at \mathbf{m} we mean the Laurent series expansion of $\zeta_r(\mathbf{s})$ at \mathbf{m} with the restriction $s_{a_1} = \dots = s_{a_h} = s$.

If we add the above restriction, then the coefficients of the Laurent series can be given in the following simpler form than Theorem 1.3.

Theorem 4.2. *Let r, m_1, \dots, m_r be positive integers. The coefficients in the restricted Laurent series of $\zeta_r(\mathbf{s})$ at $\mathbf{s} = \mathbf{m}$ can be given explicitly by using γ_n and $\zeta_k^{(l_1, \dots, l_k)}(q_1, \dots, q_k)$ for $n \geq 0$, $1 \leq k \leq r$, $(l_1, \dots, l_k) \in (\mathbb{Z}_{\geq 0})^k$ and $(q_1, \dots, q_k) \in \mathcal{D}_k \cap (\mathbb{Z}_{\geq 1})^k$.*

(Proof of Theorem 4.2.) The proof of Theorem 4.2 is quite similar to the proof of Theorem 1.3, so we omit the details. The main difference is the last step of induction, i.e. the case $\mathbf{m} = (1, \dots, 1)$. In this step, we used (1.5) in the proof of Theorem 1.3. However, in the proof of Theorem 4.2, we can not use (1.5), since its coefficients include $\gamma_{(n_1, \dots, n_k)}$. Therefore, we have to use a different method.

The idea is to use (2.4) once more. In this case, since $s_1 = \dots = s_r = s$, we can simplify (2.4) as follows:

$$\begin{aligned} \zeta_{r-1}(s, \dots, s)\zeta(s) &= r\zeta_r(s, \dots, s) \\ &+ \zeta_{r-1}(s, \dots, s, 2s) + \zeta_{r-1}(s, \dots, s, 2s, s) + \dots + \zeta_{r-1}(2s, s, \dots, s). \end{aligned}$$

The first term of the right-hand side is that we want to expand, and other terms can be expanded by assumption. Hence we obtain Theorem 4.2. \square

From (1.5) and Theorem 4.2, we can easily deduce the following corollary.

Corollary 4.3. *For any non-negative integer N , the sum*

$$\sum_{\substack{n_1 + \dots + n_r = N \\ n_1, \dots, n_r \geq 0}} \gamma_{(n_1, \dots, n_r)}$$

can be written explicitly by using γ_n and $\zeta_k^{(l_1, \dots, l_k)}(q_1, \dots, q_k)$ for $n \geq 0$, $1 \leq k \leq r$, $(l_1, \dots, l_k) \in (\mathbb{Z}_{\geq 0})^k$ and $(q_1, \dots, q_k) \in \mathcal{D}_k \cap (\mathbb{Z}_{\geq 1})^k$.

(Proof of Corollary 4.3.) By (1.5), we have

$$\begin{aligned} &\zeta_r(s, \dots, s) \\ &= \frac{1}{r!} \frac{1}{(s-1)^r} + \frac{1}{(r-1)!} \frac{1}{(s-1)^{r-1}} \sum_{n_1=0}^{\infty} \dots \sum_{n_r=0}^{\infty} \gamma_{(n_1, \dots, n_r)} (s-1)^{n_1 + \dots + n_r} \\ &= \frac{1}{r!} \frac{1}{(s-1)^r} + \frac{1}{(r-1)!} \frac{1}{(s-1)^{r-1}} \sum_{N=0}^{\infty} \sum_{\substack{n_1 + \dots + n_r = N \\ n_1, \dots, n_r \geq 0}} \gamma_{(n_1, \dots, n_r)} (s-1)^N. \end{aligned}$$

The assertion of the corollary follows from Theorem 4.2, because the above summation is the restricted Laurent series expansion at $(1, \dots, 1)$. \square

Remark 4.4. In the present paper we define multiple Stieltjes constants by (1.5), but it is not clear whether this is the most suitable definition, or not. Here we mention another definition of double Euler constant $\gamma_2(s_1)$ ($\Re s_1 > 1$), due to [10, Section 5].

In [10], $\gamma_2(s_1)$ is defined as a double analogue of the classical infinite series definition of γ , but according to [10, Proposition 5.1],

$$(4.1) \quad \gamma_2(s_1) = \lim_{s_2 \rightarrow 1} \left(\zeta_2(s_1, s_2) - \frac{\zeta(s_1)}{s_2 - 1} \right)$$

(and also $\gamma_2(s_1) = \zeta(s_1)\gamma - \zeta_2(1, s_1) - \zeta(s_1 + 1)$) holds. On the other hand, when $r = 2$, (1.5) implies

$$\zeta_2(s_1, s_2) = \frac{1}{s_2 - 1} \left\{ \frac{1}{s_1 + s_2 - 2} + \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \gamma_{(n_1, n_2)} (s_1 - 1)^{n_1} (s_2 - 1)^{n_2} \right\}$$

around the point $(1, 1)$. When $|s_2 - 1| < |s_1 - 1|$, substituting the expansion

$$\frac{1}{s_1 + s_2 - 2} = \frac{1}{s_1 - 1} \sum_{n_2=0}^{\infty} \left(-\frac{s_2 - 1}{s_1 - 1} \right)^{n_2}$$

into the above, we have

$$(4.2) \quad \zeta_2(s_1, s_2) = \frac{1}{s_2 - 1} \sum_{n_2=0}^{\infty} A_{n_2}(s_1) (s_2 - 1)^{n_2},$$

where

$$(4.3) \quad A_{n_2}(s_1) = \frac{(-1)^{n_2}}{(s_1 - 1)^{n_2+1}} + \sum_{n_1=0}^{\infty} \gamma_{(n_1, n_2)} (s_1 - 1)^{n_1}.$$

From (4.2) we see that $(s_2 - 1)\zeta_2(s_1, s_2)$ tends to $A_0(s_1)$ as $s_2 \rightarrow 1$, but from (4.1) we also see that the same limit tends to $\zeta(s_1)$. That is, $A_0(s_1) = \zeta(s_1)$. (In particular, from the case $n_2 = 0$ of (4.3) we find that $\gamma_{(n_1, 0)} = \gamma_{n_1}$.) Therefore (4.2) can be rewritten as

$$\zeta_2(s_1, s_2) - \frac{\zeta(s_1)}{s_2 - 1} = \frac{1}{s_2 - 1} \sum_{n_2=1}^{\infty} A_{n_2}(s_1) (s_2 - 1)^{n_2},$$

so with (4.1) we obtain $\gamma_2(s_1) = A_1(s_1)$.

5. THE LAURENT SERIES EXPANSION AT OTHER INTEGER POINTS

In this section, we consider the Laurent series at $\mathbf{m} \in \mathbb{Z}^r \setminus (\mathbb{Z}_{\geq 1})^r$. It is difficult to treat these points, since the harmonic product does not work well. Therefore we only use the key formula (2.1). By Lemma 3.2 (i), the integral in (2.1) is analytic at \mathbf{m} when we choose $M = M_r(\mathbf{m}) + 1$. Therefore, when \mathbf{s} is close to \mathbf{m} , we have

$$(5.1) \quad \zeta_r(s_1, \dots, s_r)$$

$$\begin{aligned}
&= \frac{1}{s_r - 1} \zeta_{r-1}(s_1, \dots, s_{r-2}, s_{r-1} + s_r - 1) \\
&\quad + \sum_{k_r=0}^{M_r(\mathbf{m})} \binom{-s_r}{k_r} \zeta_{r-1}(s_1, \dots, s_{r-2}, s_{r-1} + s_r + k_r) \zeta(-k_r) \\
&\quad + \frac{1}{\Gamma(s_r)} I(s_1, \dots, s_r; M_r(\mathbf{m}) + 1 - \eta).
\end{aligned}$$

Since the integral term $I(s_1, \dots, s_r; M_r(\mathbf{m}) + 1 - \eta)$ is analytic at \mathbf{m} , we have

$$\begin{aligned}
&\frac{1}{\Gamma(s_r)} I(s_1, \dots, s_r; M_r(\mathbf{m}) + 1 - \eta) \\
(5.2) \quad &= \sum_{n_1=0}^{\infty} \cdots \sum_{n_r=0}^{\infty} \frac{1}{2\pi i n_1! \cdots n_r!} \int_{(M_r(\mathbf{m})+1-\eta)} F_{z_r}^{(n_1, \dots, n_r)}(\mathbf{m}) \Gamma(-z_r) \zeta(-z_r) dz_r \\
&\quad \times (s_1 - m_1)^{n_1} \cdots (s_r - m_r)^{n_r},
\end{aligned}$$

where $F_{z_l}(\mathbf{s})$ is defined by (1.7). The above change of summation and integration is possible because $\Gamma(-z_r)$ decays rapidly as $|\Im z_r| \rightarrow \infty$.

The first aim of this section is to give a proof of Theorem 1.4.

(*Proof of Theorem 1.4.*) We use induction for r . When $r = 1$, the theorem follows from (1.3). We assume that the theorem holds when the number of variables is $1, 2, \dots, r-1$, and we prove that the theorem holds for r .

In the case r , we prove that it follows from (5.1) and (5.2) that Theorem 1.4 holds. From (5.1), $\zeta_r(s_1, \dots, s_r)$ can be decomposed as a sum of three terms which contain ζ_{r-1} . The last term can be expanded by (5.2), hence we only consider the first two terms. These terms consist of binomial coefficients and $\zeta_{r-1}(s_1, \dots, s_{r-2}, s_{r-1} + s_r + k)$ for integer k . Binomial coefficients are just polynomials in s_r with rational coefficients. When $(m_1, \dots, m_{r-2}, m_{r-1} + m_r + k) \in (\mathbb{Z}_{\geq 1})^{r-1}$, we use Theorem 1.3. Then $\zeta_{r-1}(s_1, \dots, s_{r-2}, s_{r-1} + s_r + k)$ can be expanded by using the coefficients of Theorem 1.3. When $(m_1, \dots, m_{r-2}, m_{r-1} + m_r + k) \in \mathbb{Z}^{r-1} \setminus (\mathbb{Z}_{\geq 1})^{r-1}$, we use the assumption of the induction. Thus, noting the following remark, we obtain Theorem 1.4.

The remark is as follows. We expand $\zeta_{r-1}(s_1, \dots, s_{r-2}, s_{r-1} + s_r + k)$ at the point $(m_1, \dots, m_{r-2}, m_{r-1} + m_r + k)$. Then the shape of the numerator part of the expansion is

$$\sum (\text{coefficient}) (s_1 - m_1)^{a_1} \cdots (s_{r-2} - m_{r-2})^{a_{r-2}} (s_{r-1} + s_r - m_{r-1} - m_r)^{a_{r-1}}.$$

Therefore we need to modify the last factor by using the relation

$$(s_{r-1} + s_r - m_{r-1} - m_r)^{a_{r-1}} = \sum_{n=0}^{a_{r-1}} \binom{a_{r-1}}{n} (s_{r-1} - m_{r-1})^n (s_r - m_r)^{a_{r-1}-n}.$$

□

In Theorem 1.4, the coefficients are not simple, since they include integrals whose behaviour is not obvious. However, since the gamma function has poles at non-positive integers, we can estimate the last term of (5.1) by $O(|s_r - m_r|)$ for

$m_r \in \mathbb{Z}_{\leq 0}$. Therefore in this case

$$\begin{aligned} & \zeta_r(s_1, \dots, s_r) \\ &= \frac{1}{s_r - 1} \zeta_{r-1}(s_1, \dots, s_{r-2}, s_{r-1} + s_r - 1) \\ &+ \sum_{k_r=0}^{M_r(\mathbf{m})} \binom{-s_r}{k_r} \zeta_{r-1}(s_1, \dots, s_{r-2}, s_{r-1} + s_r + k_r) \zeta(-k_r) \\ &+ O(|s_r - m_r|). \end{aligned}$$

Generally, the coefficients of the Laurent series expansion of $\zeta_{r-1}(s_1, \dots, s_{r-2}, s_{r-1} + s_r + k_r)$ also include integrals, and we cannot ignore the behavior of those terms. However, in some cases, such terms do not appear. Hereafter we discuss some examples when we can, or cannot, obtain the limit value if we regard the term involving the integral $I(s_1, \dots, s_r; M_r(\mathbf{m}) + 1 - \eta)$ as an O -term. For example, in the case $r = 2$ and $m_2 \leq 0$, since ζ_1 is the Riemann zeta function, we have

$$\begin{aligned} \zeta_2(s_1, s_2) &= \frac{1}{s_2 - 1} \zeta(s_1 + s_2 - 1) \\ &+ \sum_{k_2=0}^{M_2(\mathbf{m})} \binom{-s_2}{k_2} \zeta(s_1 + s_2 + k_2) \zeta(-k_2) \\ &+ O(|s_2 - m_2|). \end{aligned}$$

This equation determines the limit value at the point $(m_1, m_2) \in \mathbb{Z} \times \mathbb{Z}_{\leq 0}$. That is, we obtain the following theorem.

Theorem 5.1. *For $(m_1, m_2) \in \mathbb{Z} \times \mathbb{Z}_{\leq 0}$ and $\varepsilon_1, \varepsilon_2 \in \mathbb{C}$, we have*

$$\begin{aligned} (5.3) \quad & \zeta_2(m_1 + \varepsilon_1, m_2 + \varepsilon_2) \\ &= \frac{1}{m_2 - 1 + \varepsilon_2} \zeta(m_1 + m_2 - 1 + \varepsilon_1 + \varepsilon_2) \\ &+ \sum_{k_2=0}^{M_2(\mathbf{m})} \binom{-m_2 - \varepsilon_2}{k_2} \zeta(-k_2) \zeta(m_1 + m_2 + k_2 + \varepsilon_1 + \varepsilon_2) \\ &+ O(|\varepsilon_2|). \end{aligned}$$

In particular, the following formula holds.

Corollary 5.2. *When $m_1 \leq 0$, $m_2 \leq 0$, we have*

$$\begin{aligned} & \zeta_2(m_1 + \varepsilon_1, m_2 + \varepsilon_2) \\ &= \frac{1}{m_2 - 1} \zeta(m_1 + m_2 - 1) \\ &+ \sum_{k_2=0}^{-m_2} \frac{(-1)^{k_2}}{k_2!} m_2(m_2 + 1) \cdots (m_2 + k_2 - 1) \zeta(-k_2) \zeta(m_1 + m_2 + k_2) \\ &+ \frac{(-1)^{1-m_1-m_2}}{(1-m_1-m_2)!} \frac{(m_2 + \varepsilon_2)(m_2 + 1 + \varepsilon_2) \cdots (-m_1 + \varepsilon_2)}{\varepsilon_1 + \varepsilon_2} \zeta(m_1 + m_2 - 1) \\ &+ O(|\varepsilon_2|) + O(|\varepsilon_1 + \varepsilon_2|). \end{aligned}$$

As mentioned in the introduction, this kind of limit values was studied by several mathematicians ([1][2][6][13][14]). The second-named author also studied such limit values in [11], under the conditions

$$(5.4) \quad \varepsilon_2 \neq 0, \varepsilon_1 + \varepsilon_2 \neq 0$$

and

$$(5.5) \quad \left| \frac{\varepsilon_1}{\varepsilon_1 + \varepsilon_2} \right| \ll 1, \quad \left| \frac{\varepsilon_2}{\varepsilon_1 + \varepsilon_2} \right| \ll 1,$$

which are weaker than the conditions assumed in the previous works.

An advantage of the above corollary is that even the condition (5.5) is not required.

(*Proof of Corollary 5.2.*) When $m_1 \leq 0, m_2 \leq 0$, the case $m_1 + m_2 = 2$ does not occur, so the first term of (5.3) is

$$(5.6) \quad \frac{1}{m_2 - 1} \zeta(m_1 + m_2 - 1) + O(|\varepsilon_2|) + O(|\varepsilon_1 + \varepsilon_2|).$$

Next we consider the second term. Since $m_1 \leq 0, m_2 \leq 0$, we see that $M_2(\mathbf{m}) = 1 - m_1 - m_2$. Noting

$$\binom{-m_2 - \varepsilon_2}{k_2} = \frac{(-1)^{k_2}}{k_2!} m_2(m_2 + 1) \cdots (m_2 + k_2 - 1) + O(|\varepsilon_2|),$$

when $k_2 \neq 1 - m_1 - m_2$ we have

$$(5.7) \quad \begin{aligned} & \binom{-m_2 - \varepsilon_2}{k_2} \zeta(-k_2) \zeta(m_1 + m_2 + k_2 + \varepsilon_1 + \varepsilon_2) \\ &= \frac{(-1)^{k_2}}{k_2!} m_2(m_2 + 1) \cdots (m_2 + k_2 - 1) \zeta(-k_2) \zeta(m_1 + m_2 + k_2) \\ &+ O(|\varepsilon_2|) + O(|\varepsilon_1 + \varepsilon_2|). \end{aligned}$$

When $k_2 = 1 - m_1 - m_2$, we have

$$\zeta(m_1 + m_2 + k_2 + \varepsilon_1 + \varepsilon_2) = \frac{1}{\varepsilon_1 + \varepsilon_2} + \gamma + O(|\varepsilon_1 + \varepsilon_2|),$$

and hence

$$(5.8) \quad \begin{aligned} & \binom{-m_2 - \varepsilon_2}{k_2} \zeta(-k_2) \zeta(m_1 + m_2 + k_2 + \varepsilon_1 + \varepsilon_2) \\ &= \binom{-m_2 - \varepsilon_2}{1 - m_1 - m_2} \zeta(m_1 + m_2 - 1) \left(\frac{1}{\varepsilon_1 + \varepsilon_2} + \gamma + O(|\varepsilon_1 + \varepsilon_2|) \right) \\ &= \frac{(-1)^{1-m_1-m_2}}{(1 - m_1 - m_2)!} \frac{(m_2 + \varepsilon_2)(m_2 + 1 + \varepsilon_2) \cdots (-m_1 + \varepsilon_2)}{\varepsilon_1 + \varepsilon_2} \zeta(m_1 + m_2 - 1) \\ &+ \frac{(-1)^{1-m_1-m_2}}{(1 - m_1 - m_2)!} m_2(m_2 + 1) \cdots (-m_1) \zeta(m_1 + m_2 - 1) \gamma \\ &+ O(|\varepsilon_2|) + O(|\varepsilon_1 + \varepsilon_2|). \end{aligned}$$

Therefore from (5.6), (5.7) and (5.8) we obtain

$$\begin{aligned} & \zeta_2(m_1 + \varepsilon_1, m_2 + \varepsilon_2) \\ &= \frac{1}{m_2 - 1} \zeta(m_1 + m_2 - 1) \end{aligned}$$

$$\begin{aligned}
& + \sum_{k_2=0}^{-m_1-m_2} \frac{(-1)^{k_2}}{k_2!} m_2(m_2+1) \cdots (m_2+k_2-1) \zeta(-k_2) \zeta(m_1+m_2+k_2) \\
& + \frac{(-1)^{1-m_1-m_2}}{(1-m_1-m_2)!} \frac{(m_2+\varepsilon_2)(m_2+1+\varepsilon_2) \cdots (-m_1+\varepsilon_2)}{\varepsilon_1+\varepsilon_2} \zeta(m_1+m_2-1) \\
& + \frac{(-1)^{1-m_1-m_2}}{(1-m_1-m_2)!} m_2(m_2+1) \cdots (-m_1) \zeta(m_1+m_2-1) \gamma \\
& + O(|\varepsilon_2|) + O(|\varepsilon_1+\varepsilon_2|).
\end{aligned}$$

However when $k_2 > -m_2$, the summands in the second term are 0, since every summand has the factor $m_2(m_2+1) \cdots (m_2+k_2-1) = 0$. Furthermore, the fourth term is also 0, since $m_2(m_2+1) \cdots (-m_1) = 0$. Therefore the assertion follows. \square

At the point $(m_1, m_2) \in \mathbb{Z}_{\leq 0} \times \mathbb{Z}$, we can also deduce the result similar to Theorem 5.1 by using the harmonic product (3.10). Hence, in the case $r = 2$, we can give the Laurent series expansion at all points $\mathbf{m} \in \mathbb{Z}^2 \setminus (\mathbb{Z}_{\geq 1})^2$ with O -terms such as $O(|s_k - m_k|)$ or $O(|s_1 + s_2 - m_1 - m_2|)$.

In the case $r = 3$, there happens the situation when we cannot evaluate the limit value, if we use the above formula including O -terms for the double zeta values. For example, for $m_3 \in \mathbb{Z}_{\leq 0}$, we have

$$\begin{aligned}
& \zeta_3(s_1, s_2, s_3) \\
& = \frac{1}{s_3 - 1} \zeta_2(s_1, s_2 + s_3 - 1) \\
& + \sum_{k_3=0}^{M_3(\mathbf{m})} \binom{-s_3}{k_3} \zeta_2(s_1, s_2 + s_3 + k_3) \zeta(-k_3) \\
& + O(|s_3 - m_3|).
\end{aligned}$$

By using the Laurent series expansion for $\zeta_2(s_1, s_2)$ with O -terms, we can give the Laurent series expansion for ζ_3 at the point $(m_1, m_2, m_3) \in \mathbb{Z}^2 \times \mathbb{Z}_{\leq 0}$. When $m_2 \leq 0$ and $m_3 > 0$, by using harmonic product

$$\begin{aligned}
(5.9) \quad \zeta_3(s_1, s_2, s_3) & = \zeta(s_3) \zeta_2(s_1, s_2) - \zeta_2(s_1 + s_3, s_2) \\
& - \zeta_3(s_1, s_3, s_2) - \zeta_2(s_1, s_2 + s_3) - \zeta_3(s_3, s_1, s_2),
\end{aligned}$$

we can give the Laurent series expansion except for $m_3 = 1$. In the case $m_3 = 1$, the first term of the right-hand side of (5.9) contains the term $O(|s_2 - m_2|/(s_3 - 1))$. Hence, in this case, we can not give the limit value. Therefore, generally we can not ignore the integral terms $I(s_1, \dots, s_r; M_r(\mathbf{m}) + 1 - \eta)$ in (5.1) in order to get the limit value.

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